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ABSTRACT

This research focused on preservice elementary teachers' understanding and reconstruction of understanding about multiplication in the domain of fractions. At the start of the study, 44% of the 50 preservice teachers studied reported that they had a method for reasoning about multiplication with fractions, and 28% were able to describe a situation modeled by multiplication with a fraction operator. Although reasoning individually, the preservice teachers revealed common dimensions of understanding--about taking fractional parts of non-unit wholes and about numerical effects, referents for results, and the invariance of multiplication--as they reconceptualized multiplication with fractions. "Sense" of multiplication and "sense" of fraction relationships were forms of reasoning that supported the re-conceptualization process. (Author/MKR)

Preservice Elementary Teachers' Understanding of Multiplication Involving Fractions

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PRESERVICE ELEMENTARY TEACHERS' UNDERSTANDING OF MULTIPLICATION INVOLVING FRACTIONS

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This research focuses on preservice elementary teachers' understanding and reconstruction of understanding about multiplication in the fraction domain. At the start of the study, 44% of the 50 preservice teachers studied reported that they had a method for reasoning about multiplication with fractions; 28% were able to describe a situation modeled by multiplication with a fraction operator. Although reasoning individually, the preservice teachers revealed common dimensions of understanding — about taking fractional parts of non–unit wholes, and about numerical effects, referents for results, and the invariance of multiplication — as they reconceptualized multiplication with fractions. "Sense" of multiplication and "sense" of fraction relationships were forms of reasoning that supported the reconceptualization process.

With a renewed interest in conceptual understanding of mathematics as a fundamental purpose for classroom instruction, the conceptual understandings of teachers and prospective teachers emerge as an extremely important issue. Recent studies of teachers' knowledge of rational numbers concepts and procedures have revealed such findings as the fact that, "Ten to 25 percent of the [218] teachers missed items which we feel were at the most rudimentary level. In some cases, almost half the teachers missed very fundamental items" (Post, Harel, Behr, & Lesh, 1991, p. 186). Research into preservice teachers' understandings of multiplication with decimal numbers has resulted in similar findings (e.g., Graeber & Tirosh, 1988; Harel, Behr, Post, & Lesh, 1994).

The current research focuses on preservice teachers' understanding of — and re–construction of understanding about — multiplication with fractions. Multiplication with fractions is included in United States textbooks in the middle school grades, and teachers are expected to teach this topic with a conceptual orientation, having knowledge of its potential for modeling real world situations. This study researches prospective teachers as they enter an elementary mathematics methods course (following the conclusion of their mathematics content coursework), and addresses the following research questions: (1) How do the preservice teachers reason about multiplication with fractions as they enter the methods course? (2) What common dimensions of understanding about multiplication with fractions do the preservice teachers evidence as they construct understanding? (3) How do individual preservice teachers construct understanding? (4) What forms of reasoning appear to influence or support their re–construction of understanding about multiplication in the domain of fractions?

These questions focus on preservice teachers' understandings about multiplication with fractions from four perspectives: their entry level understandings or forms of reasoning, the content–related structures of understanding that they collectively reveal as they construct new understanding, perspectives on individual preservice teachers' methods of constructing and re–constructing understanding, and theoretical speculation about reasoning that supports the re–construction of understanding about multiplication in the fraction domain. The theory, or theoretical framework, within which this research is conducted is the research–based theory that whole number multiplication must be re–conceptualized in the rational number domain (Greer, 1994; Hiebert & Behr, 1988):

> It is likely that there are not smooth continuous paths from early addition and subtraction to multiplication and division, nor from whole numbers to rational numbers. Multiplication is not simply repeated addition, and rational numbers are not simply ordered pairs of whole numbers. The new concepts are not the sums of previous ones. Competence with middle school number concepts requires a break with simpler concepts of the past, and a reconceptualization of numbers itself. (Hiebert et al., p. 8)

The research questions are framed within this theory. It was the purpose of this research to infer prospective elementary teachers' understandings about multiplication with fractions, their methods of constructing and re–constructing understanding,[1] common structural dimensions in their re–construction of understanding, and particular forms of reasoning supporting their re–conceptualization of multiplication in the fraction domain.

Methodology

Fifty preservice elementary teachers enrolled in two sections of a required state university mathematics methods course taught by the researcher contributed research data in a three–phase qualitative research design:

Phase I (weeks 1–4): Entry–level assessment of understandings of preservice teachers through one–hour individual audiotaped interviews focused on their work on a written inventory that requested them to create and solve a word problem modeled by each of four given fraction and whole number multiplication expressions.

Phase II (weeks 5–10): Instruction about multiplication with whole numbers and fractions—conceptual models for multiplication, numerical patterns in multiplication with whole numbers and fractions, and situations modeled by multiplication—in order to support the preservice teachers in construction and re–construction of understanding (rather than to provide and measure the effects of a treatment); collection of coursework; and keeping of field notes of the class sessions about multiplication.

Phase III (weeks 11–16): Inferring of understandings and construction processes in the preservice teachers' development of understanding, through one–hour audiotaped individual interviews during which they were asked to describe their understandings about multiplication with fractions and to conceptually inter-

[1] Students both constructed understanding about whole number multiplication and re–constructed understanding about whole number multiplication in the fraction domain.

pret a sample of fraction multiplication expressions, through continuing coursework, and through class field notes.

During and following Phases I–III, the data were interpreted and analyzed to address each research question. Processes of analysis, including interpreting, inferring, and categorizing forms of reasoning and structures of understanding, were utilized. Review and evaluation of the interpretations and findings by a cohort of mathematics education researcher experts, and triangulation of the data through the complementary data sources for each preservice teacher, were utilized.

Findings

As they entered the methods course, 8 (16%) of the 50 preservice teachers were able to create word problems modeled by all four of the whole number and fraction multiplication expressions on the written Inventory [24×37, $7 \times \frac{1}{4}$, $1\frac{1}{2} \times \frac{1}{3}$ and $\frac{2}{3} \times \frac{3}{4}$]. Eighteen students[2] (36%) were not able to create a word problem for any of the three fraction multiplication expressions on the Inventory, and an additional 18 students (36%) were able to create a word problem for only one of the three fraction multiplication expressions, the expression with one whole number factor [$7 \times \frac{1}{4}$]. In other words, 36 students, 72% of the students in the study, entered the methods course unable to describe a situation that would be modeled by multiplication with a fraction operator. In addition 4 students entered the methods course unable to construct a word problem appropriately modeled by the whole number multiplication expression 24×37.

Twenty–two students (44%) entered the methods course reporting that they had been taught or had discovered a form of reasoning about multiplication with fractions less than 1 [the forms being that multiplying by a fraction less than 1 reduces other numbers, divides other numbers, or takes a fraction <u>of</u> other numbers]. Each of the 14 students who succeeded in creating a word problem for one or both of the Inventory multiplication expressions with a fraction operator was among this number. The other 28 students in the study (56%) reported that they had no method for reasoning about multiplication with fractions as they entered the methods course. Their knowledge was strictly procedural.

Structural Dimensions of Understanding

Structural dimensions or benchmarks of learning common to the preservice teachers as they constructed and re-constructed understanding about multiplication during the study are described as follows:

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[2] The preservice teachers are referred to alternatively as "preservice teachers" or "students," They were students in the methods course in which this study was conducted.

Understanding the Numerical Effects of Fraction Multiplication

Students provided evidence of constructing knowledge about the numerical results of multiplying with fractions: that multiplying by a number less than and greater than 1, respectively, reduces or enlarges other numbers. During their second interviews, 48 of the 50 preservice teachers discussed the numerical results of fraction multiplication in relation to the multiplicative identify, 1 (The remaining two students were confused in their understanding.). However, not all students could interpret fraction multiplication expressions using the commutative property, or in terms of the influence of each factor in the expression on the other factor; fewer than 50% of the preservice teachers departed the study demonstrating the flexibility to interpret expressions in two directions.

Conceptualizing a Fractional Part of a Non–Unit Whole

For one–half of the students, conceptualizing a fractional part of a quantity other than one discrete unit (other than 2/3 of 1 cup or 2/3 of 1 hour, for example), involved new learning. Even though 14 of the students in the study had learned or discovered that they could reason about multiplying with fractions using the word <u>of</u> (e.g., 1/3 <u>of</u> 6 for 1/3 x 6), 10 of these students, in addition to other students in the study, experienced difficulty physically representing and making sense of expressions such as "2/3 *of* 3/4" (2/3 x 3/4). Learning to operate on a non–unit quantity when the operator is a fraction less than 1 (such as learning to take 2/3 <u>of</u> 3/4 rather than 2/3 of 1 unit) emerged as an important benchmark in the preservice teachers' continuing development of understanding of multiplication with fractions. As students constructed understanding, some students experienced difficulty, similarly, conceptualizing a fraction greater than 1 operating on a non–unit whole, such as $1\frac{1}{3} \times \frac{3}{4}$ or $2\frac{1}{2} \times 4$. All students revealed during their second interviews that they had constructed understanding, in some form, of the taking of a fractional part of a non–unit whole. Methods students used to conceptualize this process with operators less than and greater than 1 differed.

Interpreting Referents of Results

Approximately one–third of the students experienced difficulty identifying the referents, or units of measure, for fraction multiplication results. They experienced difficulty understanding the whole (or referent unit of measure) to which the result of the multiplication referred. Most commonly, students who experienced difficulty interpreted the referent for the result of multiplication using the original quantity being reduced or enlarged: 3/4 x 1/2 = 3/8, for example, was interpreted as 3/8 of "1/2 unit" rather than "3/8 unit." Some difficulties in interpreting referents seemed related to the fact that students attempted to make sense of fraction multiplication through representing the numerical answers to, rather than the conceptual processes involved in, fraction multiplication expressions — such as in attempting to represent the answer of 3/8 (and interpreting this as 3/8 within the 1/

2 unit), rather than representing the conceptual process of taking 3/4 of 1/2 and then interpreting the referent for the 3/8. Reasoning about and representing the conceptual processes involved in fraction multiplication expressions supported students in interpreting referents for multiplication results.

Understanding of Multiplication as an Invariant Process

One–third of the students described multiplication, during their second interviews, evidencing a conceptualization of multiplication as an invariant process—modeling the same situations or illustrating the same conceptual models whether with whole numbers or fractions. Other students, who could interpret multiplication expressions with fraction and whole number operators, interpreted as distinctly different the concepts or models involving whole number operators and those involving fraction operators—frequently a repeated addition or equal groups model for whole numbers and a "breaking down" concept (in the words of several students) for fraction operators. Conceptualizing multiplication as an operation modeling the same concepts or situations with whole number and fraction operators was difficult. Although students could discuss the numerical effects of multiplying by numbers greater than and less than 1, interpreting multiplication with operators both greater than and less than 1 using the same conceptual model (e.g., equal groups, multiplicative compare, and area) was difficult.

Individual Forms of Reasoning and Constructing Understanding

The construction processes revealed by preservice teachers as they developed understanding of the four structural dimensions described above were distinctly unique (see Azim, 1995). During their second interviews, students' reasoning and understandings could be categorized in three categories: students constructing concepts for the first time during the interview—19 students (38%); students re-constructing concepts—17 students (34%); students building more complex constructions on their own—14 students (28%).

Forms of Reasoning Supporting Subjects' Construction of Understanding

Two particular forms of reasoning seemed to support students' construction of understanding of multiplication with fractions throughout the study: (1) their multiplication "sense," or forms of reasoning about multiplication (particularly with whole numbers), and (2) their fraction "sense," or sense of size relationships between fractions or fractional quantities. Students who had a more clearly developed sense of multiplication with whole numbers (having one or two even implicitly constructed conceptual models to draw on) and who could discuss fractional quantities in relation to each other (such as the relationship that 1/2 is one–third of, or one of 3 equal parts in, $1\frac{1}{2}$), drew on these senses to construct meaning for fraction multiplication. Students who demonstrated a limited understanding of multiplication with whole numbers (having a very weak or no concept of this operation), and a limited sense of fraction relationships, experienced greater difficulty interpreting fraction multiplication. The most powerful finding of this study was the repeated observation of the influence of the interaction of the levels of development of these two forms of reasoning on the preservice teacher's re–construction of understanding about multiplication. This finding is parallel to a summary by Sowder (1992) regarding good *estimators*: "They demonstrate a deep understanding of numbers and operations, and they continually draw upon that understanding" (p. 375). Deep understanding of multiplication and of fractions supported preservice teachers in their construction of understanding about fraction multiplication.

Other Theoretical Connections

The data in this study support Greer's (1994, p. 77) observation that, "The invariance of multiplication... over the numbers is a powerful idea that potentially can be harnessed to overcome the limitations of intuition." Students constructed understanding (and started to construct understanding) about multiplication as an invariant operation through different reasoning processes; knowledge that multiplication is invariant supported them in their attempts to re–interpret multiplication with fractions. Students also revealed evidence of using both quantitative reasoning (reasoning about quantities without numerical reference) and numerical reasoning (reasoning about numbers evaluating quantities)—two forms of reasoning theorized by Thompson (1994)—in their re–construction of understanding. Students who constructed each of the four dimensions (described above) to greater degrees evidenced both numerical and quantitative reasoning in the construction process.

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